

Pure Graph of a Commutative Ring

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Abstract

A new definition of a graph called Pure graph of a ring denote $Pur(R)$ was presented , where the vertices of the graph represent the elements of R such that there is an edge between the two vertices α and β if and only if $\alpha = \alpha\beta$ or $\beta = \beta\alpha$, denoted by $pur(R)$. In this work we studied some new properties of $pur(R)$ finally we defined the complement of $pur(R)$ and studied some of its properties .

Keywords: Graph theory, commutative ring.

1- Introduction

There is a lot of research linking between graph theory and algebraic ring theory. Ali Majidinya et.al. studied Ring in which the annihilator of an ideal is pure [1]. Bhavanari S. etal defined Prime Graph of a Ring [3] Mohammad Habibi etal. They studied clean graph of a ring [7]. Dhiren K.Basnet and Jayanta Bhattacharyya defined nil clean graph of rings [4], Jafari A. and Sahebi S., studied Vonneumann regular graphs associated with rings[6], A graph G is defined by an ordered pair $(V(G), E(G))$, where $V(G)$ is a nonempty set whose elements are called vertices and $E(G)$ is a set (may be empty) of unordered pairs of distinct vertices of $V(G)$. the element of $E(G)$ are called edges of the graph G . we denote by $\overline{\alpha\beta}$, an edge between two end vertices α and β [8] .

In this paper we give new definition named Pure graph of ring and denoted by $pur(R)$ with some properties of this new graph .

Basic concept :

Definition 1.1:[1] An element p in R is called pure element if there exist q in R such that $p=pq$.

Definition 1.2:[2] Let H be a graph, $V(G)$ the set of vertices of G and $S \subseteq V(H)$, the set S is said to be a dominating set if the following condition is satisfy ; $a \in V(H)$ implies either $a \in S$ or there exists $k \in S$ such that a and k are adjacent .

Definition 1.3:[8] A cycle graph with n vertices denoted by C_n , obtained by joining the two end vertices of a path graph and then each vertex of a cycle have degree two .

Definition 1.4:[5] The complete tripartite graph $K_{1,1,p}$. It is a graph consisting of p triangles sharing a common edge is called triangular book.

Theorem 1.5:[2] A connected graph G is Euler if and only if its edge set can be decomposed into cycles.

2- Main Result :

Definition 2.1: let R be a ring. A graph $K(V, E)$ where $V(K) = R$ and $E(K) = \{ \overline{\alpha\beta} / \alpha = \alpha\beta \text{ or } \beta = \beta\alpha \text{ and } \alpha \neq \beta \}$ is called Pure graph of R and denoted by $pur(R)$

Example :

$$Z_2 = \{0, 1\}$$

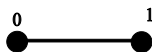


Fig 1: $pur(Z_2)$

$$Z_3 = \{0, 1, 2\}$$

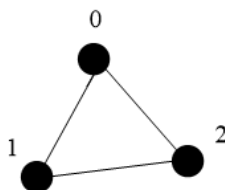


Fig 2: $pur(Z_3)$

$$Z_4 = \{0, 1, 2, 3\}$$

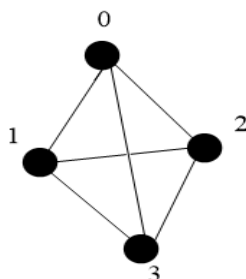


Fig 3:*pur*(Z_4)

$Z_5=\{0,1,2,3,4\}$

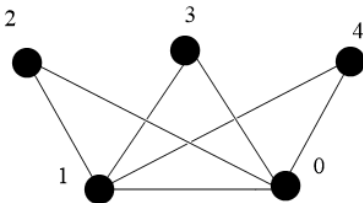


Fig 4:*pur*(Z_5)

$Z_6=\{0,1,2,3,4,5\}$

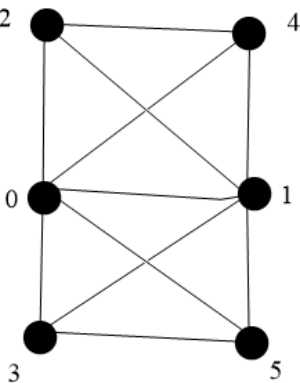


Fig 5: *pur*(Z_6)

$Z_7=\{0,1,2,3,4,5,6\}$

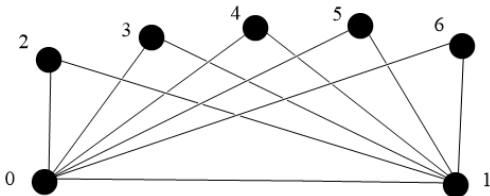
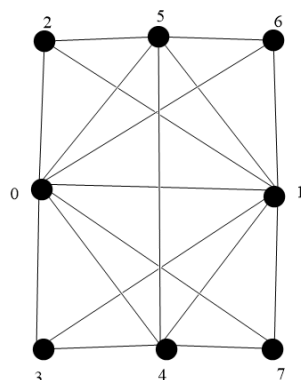


Fig 6: *pur*(Z_7)

$Z_8=\{0,1,2,3,4,5,6,7\}$

Fig 7: $pur(Z_8)$

Remarks 2.2: Let $Pur(R)$ be Pure graph where $R = Z_n$ then

- 1- $Pur(R)$ has no self loops
- 2- Since $0 = 0\mu$ and $\mu = \mu 1$ for all $0 \neq \mu \neq 1 \in R$ there is an edge from 0 and 1 to μ for all $\mu \in V(G) = R$ so $\text{degree}(0) = \text{degree}(1) = |R| - 1$
- 3- For any two non-zero elements a, b in R there are edge one from 0 and 1 to a and another edge from 0 and 1 to b this show that the graph $pur(R)$ is connected graph. $d(0, a) = d(a, 1) = 1$ and $d(a, b) \leq 2$ for any two non-zero elements $a, b \in R$
- 4- If there are two non-zero elements a, b in R such that $a = ab$ or $b = ba$, then the subgraph produced by $\{0, 1, a, b\}$ is K_4 graph, note that the graph $Pur(R)$ where $R = Z_6$ as fig 5, Sub graph produced by $\{0, 1, 2, 4\}$ is K_4 .
- 5- If $R = Z_n$ then $\max Pur(Z_n) = n-1$ and $\min Pur(Z_n) \geq 2$.

Remark 2.3:

- 1- $v_1 = v_1 v_2$ or $v_2 = v_2 v_1$ if and only if the distance between v_1 and v_2 equal 1.

Proof:

Suppose that $v_1 = v_1 v_2$ or $v_2 = v_2 v_1$ and $v_1 \neq v_2 \neq 0$ or 1 then $v_1 v_2 \in E(Pur(R))$ and so by definition of $Pur(R)$ then $d(v_1, v_2) = 1$.

Conversely, suppose $d(v_1, v_2) = 1$, if $(v_1 = 0 \text{ or } v_2 = 0)$ or $(v_1 = 1 \text{ or } v_2 = 1)$ then $v_1 = v_1 v_2$ or $v_2 = v_2 v_1$

if $d(v_1, v_2) = 1$ and $v_1 \neq v_2 \neq 0$ or 1 then $v_1 v_2 \in E(Pur(R))$ which implies

$$v_1 = v_1 v_2 \text{ or } v_2 = v_2 v_1$$

- 2- $u_1 \neq u_1 u_2$ or $u_2 \neq u_2 u_1$ if and only if the distance between u_1 and u_2 equal 2.

Proof:

Let $u_1 \neq u_1 u_2$ or $u_2 \neq u_2 u_1$ then there is no edge between u_1 and u_2 , so the distance between u_1 and u_2 is largest than 1. since $0 = 0u_1, 0 = 0u_2, \overline{u_1 0}, \overline{u_2 0} \in E(Pur(R))$, hence the distance between u_1 and u_2 equal 2.

Conversely, let the distance between u_1 and u_2 equal 2 since $d(u_1, u_2) \neq 1$, there is no edge between u_1 and u_2 so $u_1 \neq u_1 u_2$ or $u_2 \neq u_2 u_1$

Theorem 2.4 : If $R = Z_p$, and $p \geq 3$, p (prime number), then $Pur(R)$ is a triangular book graph.

Proof :

It is clear that 0, 1 adjacent to all remaining vertices in $Pur(Z_p)$ by definition and there is no edge between any other two vertices α and β where (α and $\beta \neq 0$ or α and $\beta \neq 1$) since $\alpha \neq \alpha\beta \mod(p)$ or $\beta \neq \beta\alpha \mod(p)$.

Theorem 2.5: let $R = Z_p$, and $p \geq 3$ (p is prime number), then $Pur(R)$ has $p - 2$ of cycle C_3 .

Proof:

By definition of pure graph of a ring it is clear that 0 and 1 adjacent to all ramming vertices then $\forall a \in V(Pur(Z_p)), 0 \neq a \neq 1$ then we have a cycle of length 3 $\{0, 1, a\}$, that is the number of cycle C_3 is $p-2$.

Theorem 2.6: If $R = Z_p$, p is prime number then $Pur(Z_p)$ is Euler graph.

Proof :

By theorem (2.5) the graph $Pur(Z_p)$ that is the set edges can be decomposed into cycles then $Pur(Z_p)$ is Euler graph by theorem (1.5)

Theorem 2.7 : If $R = Z_p$, and $p \geq 3$ (p is prime number), then $Pur(R)$ has $\sum_{i=3}^p (p - i)$ of C_4

Proof :

Suppose that $0 \neq v_1 \neq 1$ be a vertex in $Pur(R)$ then we have $(p-3)$ of C_4 start from the vertex v_1 where $p-3$ is the number of ramming vertices, now we take another vertex v_2 it is clear that is the number of ramming vertices is $p-4$ then we have $(p-4)$ of C_4 start from the vertex v_2 . Repeat the process for the rest of the vertices that is we have

$$(p-3) + (p-4) + \dots + 1 = \sum_{i=3}^{p-1} (p-i).$$

Corollary 2.8: The graph $Pur(Z_n)$, $n > 3$ not prime number has at least one of K_4 .

Proof :

By definition of $Pur(R)$ for any two vertices $0 \neq v_1, v_2 \neq 1$ we have a cycle $C_4 \{0, v_1, 1, v_2\}$ and 0 and 1 are adjacent that is if $v_1 = v_1 v_2$ or $v_2 = v_2 v_1$ then we have K_4 sub

graph of $Pur(Z_n)$, since n not prime then there are another vertices so by definition of $Pur(R)$ has another K_4 .

3- Invariants of Pure graph:

In this part, we studied some results related to invariants of graph theory. The girth of $Pur(R)$ is compute in the following theorem.

3.1 Girth of $Pur(R)$

In a graph G , the girth of G is the length of the shortest cycle in G . We have following results on girth of $Pur(R)$.

Theorem 3.1.1 : If $R = Z_n$, and $n \geq 3$, then the Girth of $Pur(R)$ is equal to 3.

Proof :

It is clear that ,Since 0 and 1 adjacent to all remaining vertices in $Pur(R)$ and also 0 and 1 are adjacent to other , that is the shortest cycle in $Pur(R)$ is of length 3

3.2 Dominating set of Pure graph :

Let G be a graph, a subset $S \subseteq V(G)$ is said to be dominating set for G if for all $x \in V(G)$, $x \in S$ or there exists $y \in S$ such that x is adjacent to y . Following theorem shows that for a finite commutative ring dominating number is 1, where dominating number is the cardinality of smallest dominating set.

Theorem 3.2.1 : The dominating number of $Pur(Z_n)$ is 1.

Proof :

Since the smallest dominating set in $Pur(Z_n)$ graph is $\{0\}$ and $\{1\}$ because 0 and 1 are adjacent to all vertices in $Pur(Z_n)$ graph then the dominating number is 1.

4- The complement of $pur(R)$.

Definition 4.1: let R be a ring. A graph $Pur^c(R)$ is said to be the complement of $pur(R)$ where the vertex set is the ring R and the edge set equal to $\{ \overline{\alpha\beta}/\alpha \neq \beta \text{ or } \beta \neq \alpha \text{ and } \alpha \neq \beta \}$.

Example : Consider Z_n (the ring of integers modulo n).

- 1- Where $n = 2, 3, 4$ $Pur^c(Z_n)$ is empty graph since, where $n=2$ then $V(Pur(Z_2)) = \{0, 1\}$ and $E(Pur(Z_2)) = \{\overline{01}\}$ since $01=0$, and hence there no edges in $Pur^c(Z_2)$

,also where $n=3$ then $V(Pur(Z_3)) = \{0,1,2\}$ and $E(Pur(Z_3)) = \{\overline{01}, \overline{02}, \overline{12}\}$ since $01=0, 02=0, 21=2$, and hence there no edges in $Pur^c(Z_3)$, now if $n=4$ then $V(Pur(Z_4)) = \{0,1,2,3\}$ and $E(Pur(Z_4)) = \{\overline{01}, \overline{02}, \overline{03}, \overline{12}, \overline{13}, \overline{23}\}$ since $01=0, 02=0, 03=0, 21=2, 31=3, 23=2$ and hence there no edges in $Pur^c(Z_4)$.

- 2- Where $n = 5$ $Pur^c(Z_5)$ is disconnected graph since $V(Pur(Z_5)) = \{0,1,2,3,4\}$ and $E(Pur(Z_5)) = \{\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{12}, \overline{13}, \overline{14}\}$ since $01=0, 02=0, 03=0, 04=0, 21=2, 31=3$ and $41=4$ and hence the $E(Pur^c(Z_5)) = \{\overline{23}, \overline{24}, \overline{34}\}$ that is 0 and 1 don't adjacent to any vertex in $Pur^c(Z_5)$ the blew figure show that

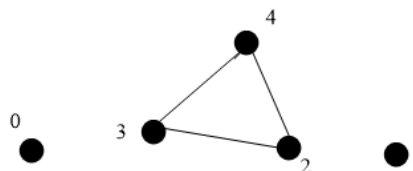


Fig 8 : $Pur^c(Z_5)$

Note: the $Pur^c(Z_n)$ is disconnected graph since 0,1 is adjacent to all vertices in $Pur(Z_n)$.

then $Pur^c(Z_n)$ has at least three components.

Theorem 4.2: A graph $Pur^c(Z_p)$, p is prime number is disconnected graph has three components two isolated vertices 0 and 1 and K_{p-2} complete graph.

Proof :

By theorem 2.4 $Pur(Z_p)$ is triangle book graph and 0,1 adjacent to all vertices, and there is no edge between any two vertices that is 0 and 1 are isolated vertices in $Pur^c(Z_p)$, now since the rest of the vertices (their number is $p-2$) are not adjacent to each other in $Pur(Z_p)$ they will be adjacent to each other in $Pur^c(Z_p)$, this means that $Pur^c(Z_p)$ contains K_{p-2} complete sub graph.

5- Conclusion :

The definition of pure graph of commutative ring $Pur(R)$ was introduced in this work and the number of cycle C_3 , C_4 and K_4 in $Pur(R)$ was found where $R = \mathbb{Z}_p$, also gave girth and the dominating number of $Pur(R)$, in addition, a definition is provided the complement of pure graph denoted by $Pur^c(R)$.

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