On - open sets in Bitopological space

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Abstract:

In this paper we define the concept of δ-open sets in bitopological space and we use them in order to study some types of continuous functions between bitopological spaces.

المستخلص:

في هذا البحث قمنا بتعريف المجاميع المفتوحة من النوع δ في bitopological space ودرسنا بعض أنواع من الدوال المستمرة بين هذه الفضاءات .

1) Introduction

The concept of bitopological space was introduced by Kelly [1]. The definition of δ-open and weakly δ-open sets in (X,τ) was introduced by Velicko in 1968, Mayada Ghassab 2003[2] respectively.

In this paper we define δ-open and weakly δ-open sets in bitopological space .If (X,τ,τ') is a bi-topological space and $A \subseteq X$, then A is δ-open iff for any $a \in A$ there is a regular open set G such that $a \in G \subseteq A$ .The complement of δ-open set is called δ-closed set. In 2003, Mayada [2] introduced and studied a new of class open sets, called weakly δ-open sets. This class of open sets is defined in (X,τ) .In this paper we study δ-open and weakly δ-open sets in (X,τ,τ') and we introduce new types of continuous functions between bitopological spaces namely δ-continuous function, δ-open function, w-δ-open, almost w-δ-open, always open function, we characterize these classes of function, study some of their basic properties and investigate their relationships to continuous functions and to other types of functions between bitopological spaces.

Definition (1.1): Let (X,τ,τ') be a bitopological space and $A \subseteq X$, then A is called regular open set if $A = int_\tau(cl_{\tau'}(A))$.

A set F is called regular closed set if $F = cl_\tau(int_{\tau'}(F))$. The family of all regular open (resp. regular closed) sets of a space X is denoted by RO (X) (resp. RF (X)).

Definition (1.2): Let (X,τ,τ') be a bitopological space and $W \subseteq X$, then W is called δ- open set iff for any $x \in W$ there exists a regular open G such that $x \in G \subseteq W$. or equivalently: W is δ- open set iff W is union of regular open sets . the complement of δ- open set is called δ-closed set.

Definition (1.3): - A subset W of X is called open set (resp. F closed set) in (X,τ,τ') if $int_\tau(int_{\tau'}(W))=W$ (resp. $cl_\tau(cl_{\tau'}(F))=F$) .i.e. W is open in (X,τ,τ') iff W is open in (X,τ) and open in (X,τ') (resp.F is closed set in (X,τ,τ')) if F is closed set in (X,τ) and closed set in (X,τ').

Remarks (1.4):

1) Every regular open set is δ-open set.
2) Every regular closed set is δ-closed set.

The converse of (1) and (2) is not true in general: the following example explains that:

Example (1.5): - let X {1, 2, 3}

$\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$

$\tau' = \{\phi, X, \{1\}, \{2,3\}\}$

Let A={1,2}

We note that A is δ-open set but it is not regular open set.Also,we note that {1, 3} is closed set but it is not δ-closed set.

Remarks (1.6): -

1) The concept of open and δ-open set are independent
2) The concept of open and regular open are independent.

The following example shows that: -

Example: -let $X=\{a, b, c\}$;

$\tau=\{\phi, X, \{a\}, \{b\}, \{a, b\}\}$

$\tau'=\{\phi, X, \{a\}\}$

It is clear that $\{a\}$ is open set but it is not regular and it is not δ-open set

In example (1.5), we note that $\{1, 2\}$ is δ-open set but it is not open set.

3) The union of two regular open sets need not be regular open set .in example (1.5), $\{1\}$, $\{2\}$ are regular open sets but $\{1, 2\}$ is not regular open set.

Theorem (1.7): - the union of δ-open sets is δ-open set.

Proof: -let $\{A_I\}$ is a family of δ-open set Let $a\in \cup A_I$ then there is A_j such that $a\in A_j$, but A_j is δ-open set then there is a regular open set G such that $a\in G\subseteq A_j$, thus $a\in G\subseteq A_j$; therefore, $\cup A_I$ is δ-open set.

Lemma (1.8): The intersection of a finite number of regular open sets is regular open set.

Theorem (1.9): The intersection of a finite number of δ-open sets is δ-open set.

Proof: -let $\{A_i\}^n_{i=1}$ be a family of δ-open sets

Let $a\in\cap^n_{i=1}A_i$ then $a\in A_i$, for any $i=1,2\ldots n$

Since A_i is δ-open set then there is a regular open set G_i such that $a\in G_i\subseteq A_j$, thus $a\in\cap^n_{i=1}G_i \subseteq\cap^n_{i=1}Ai$ is but $\cap^n_{i=1}G_i$ is regular open set, therefore $\cap^n_{i=1}Ai$ is δ-open set .

Definition (1.11): - a point x is a space X is called δ-interior (resp. δ-boundary, δ-limit) of $A \subseteq X$ iff there is a regular open neighborhood U of x in X such that $x\in U\subseteq A$ (resp.iff for any δ-open neighborhood U of x in X then $U\cap A\neq\phi$ and $U\cap(X-A) \neq\phi$,iff for any δ-open neighborhood)U of x in X then $U-\{x\}\cap A \neq\phi$) The set of all δ-interior point of A is said to be δ-interior set of A and it is denoted by δ- int A (resp. the set of all δ-boundary points of A is said to be δ-boundary set and it is denoted by δ-∂A , The set of all δ-limit points of A is said to be δ-limit A and it is denoted by δ-A$'$).

Definition (1.12):-

The intersection of all δ-closed sets which are containing W is called δ-closure of A and it is denoted byδ- cl A.

Proposition (1.13):-Let A, B be two subsets of X and $A \subseteq B$, then

1. δ- int A is δ-open set.
2. δ- int $A\subseteq A$
3. δ- int $A\subseteq$ δ- int B.
4. δ-$A'\subseteq$ δ-B' .
5. δ-cl $A \subseteq$ δ-cl B
6. δ- int A= Ext_δ (X-A)
7. **δ-∂A= δ-∂(X-A)**

Theorem (1.14):

Let (X,τ,τ') be a bitopological space , let A,B be two subsets of X , then :

1. δ- int $A\cup$ δ- int $B \subseteq$ δ- int$(A\cup B)$.
2. δ- int $(A \cap B)$ = δ- int $A \cap$ δ- int B
3. δ-$A'\cup$δ-B' = δ-$(A\cup B)'$
4. δ-$(A \cap B)' \subseteq$ δ- $A' \cap$δ-B'

2) Some types of continuous and open functions between bitopological spaces:

Definition (2.1) :.A function $f : (X, \tau,\tau') \to (Y,\sigma,\sigma')$ is said to be continuous function (resp. (δ-continuous)) if $f^{-1}(U)$ is open set (resp. δ-open) , for every open set U (resp. δ-open) set in Y .

Remark (2.2):

The concept of continuity and δ-continuity are independent.

Example: Let X=Y= {1, 2, 3}

$\tau = \{ \phi , X, \{3\}, \{2,3\} \}$

$\tau' = \{ \phi , X, \{1\} ,\{ 2,3\} \}$

$\sigma = \{\phi, Y, \{1\}, \{2, 3\}\}$

$\sigma' = \{\phi, Y, \{2, 3\}\}$

Let $f: (X,\tau,\tau') \to (Y,\sigma,\sigma')$ be the identity function ,we note that f is continuous function but it is not δ-continuous function

Example:

Let X ={1,2,3}=Y

$\tau = \{ \phi , X ,\{1\} ,\{ 1,2\} \}$; $\tau' = \{ \phi ,X ,\{1\} ,\{ 2\} ,\{1,2\}\}$

$\sigma = \{\phi, Y,\{1\},\{2,3\}\}$; $\sigma' = \{\phi, Y ,\{2,3\}\}$

let $f: (X, \tau,\tau') \to (Y,\sigma,\sigma')$ be the identity function , f is δ-continuous function but it is not continuous function.

Theorem (2.3): let $f: (X, \tau,\tau') \to (Y,\sigma,\sigma')$ be the function then the following statements function

1. f is δ-continuous function
2. for any δ-closed set F in Y then $f^{-1}(F)$ is δ-closed set in X
3. for any $A \subseteq X$ then $f(\delta\text{-cl } A) \subseteq \delta\text{-cl } (f(A))$
4. for any $B \subseteq Y$ then $\delta\text{-cl } (f^{-1}(B)) \subseteq f^{-1}(\delta\text{-cl } B)$.

Proof: (1) $\to$(2)

Let $f:(X,\tau,\tau') \to (Y,\sigma,\sigma')$ be a δ-continuous function

Let F be δ-closed set in Y then X-F is δ-open set in Y

But f is δ-continuous function so f^{1} (X-F) is δ-open set in X

Thus X- f^{1} (X-F) is δ-closed set in X.

Since X- f^{1} (X-F) = f^{1} (X); therefore, f^{1} (F) is δ-closed set in X.

Proof: (2) $\to$(3)

Let $A \subset X$

Since $f (A) \subseteq \delta\text{-cl } (f (A))$ then $f^{-1} (f (A)) \subseteq f^{-1} (\delta\text{-cl } (f (A)))$

Since $A \subseteq f^{-1} (f(A))$so $\delta\text{-cl } A \subseteq f^{-1} (\delta\text{-cl } (f(A))$ then $f(\delta\text{-cl } A) \subseteq f(f^{-1} (\delta\text{-cl } (f(A))$, but $f(f^{-1} (\delta\text{-cl } (f(A)) \subseteq \delta\text{-cl } (f(A))$ therefore , $f(\delta\text{-cl } A) \subseteq \delta\text{-cl } (f(A))$.

(3)$\to$(4)

Proof: Let $B \subseteq Y$

By (3) we get $f (f^{-1} (\delta\text{-cl } (f (B)) \subseteq \delta\text{-cl } (f(A))$

Since $f (f^{-1} (B)) \subseteq B$ then by proposition (1.13), we get

$\delta\text{-cl } (f (f^{-1} (B)) \subseteq \delta\text{-cl } (B)$

Since $f (\delta\text{-cl } (f^{-1} (B))) \subseteq \delta\text{-cl } (f (f^{-1} (B))$ then $f^{-1} (f(\delta\text{-cl } (f^{-1} (B))) \subseteq (\delta\text{-cl } (f(f^{-1} (B)))$ thusδ- cl $(f^{-1} (B)) \subseteq f^{-1}(\delta\text{-cl } (B))$.

(4) $\to$(1)

Proof: let U be δ-open set in Y then X-U is δ-closed set so X-U =δ-cl (X-U)

To prove f^{1} (U) is δ-open set in X, we most prove f^{1} (U)= δ-cl (X-f^{1} (U))

Since X-U = δ-cl (X-U) then f^{1} (X-U) = f^{1} (δ-cl (X-U)) thus

X-f^{1}(X-U) = X- f^{1}(δ-cl (X-U)).

By (4) we know that: δ-cl (X- f^{-1} (U)) $\subseteq$ f^{-1}(δ-cl (X-U)) and

f^{1}(δ-cl (X-U)) $\subseteq$ f^{1}(X-U) $\subseteq$ δ-cl f^{1}(X-U)), thus we get δ-cl (X-f^{1} (U)) $\subseteq$ X- f^{1} (U); therefore f^{1} (U) is δ-open set in X.

Definition (2.4): A point x in X is called δ-adherent point of a set A iff every regular open set containing x has non empty intersection with A .the set of all δ-adherent by δ-cl A is called the δ-closure of A and it is denoted by δ-cl A. and a subset A of a space X is called weakly δ-open set if there is an open set O such that δ-cl A= δ-cl O

Definition (2.5):.A function f : $(X,\tau,\tau')\to(Y,\sigma,\sigma')$ is said to be weakly-δ- continuous function (resp. (super-continuous, w-super continuous) function iff for any weakly δ-open (resp. open ,open set)V in Y then $f^{-1}(V)$ is weakly δ- open set (resp. δ-open ,weakly δ-open set) in X .

Definition (2.6):.A function f : $(X,\tau,\tau')\to(Y,\sigma,\sigma')$ is said to be δ-open (resp. δ-closed ,g δ-closed ,w- δ-open ,almost w- δ-open ,always δ-open) if f(U) is δ-open(δ-closed , g δ-closed ,w-δ-open, open) set in Y ; for every (resp. g δ-closed ,w-δ-open ,open,w-δ-open) set U in X .

Theorem (2.7): A function f: $(X,\tau,\tau')\to(Y,\sigma,\sigma')$ is δ-open function iff f (int A) $\subseteq$ δ-int (f(A)); for any A$\subseteq$ X.
Proof: to prove f (δ-int A) $\subseteq$ δ-int (f (A)); for any A$\subseteq$ X.

Since δ-int A$\subseteq$A then f(δ-int A) $\subseteq$f(A) so δ-int (f(δ-int (A)) $\subseteq$ int_δ (f(A)) since δ-int (A) is δ-open set and f is δ-open function then f(δ-int A) is δ-open in Y ,thus f(δ-int A)= f(δ-int A);therefore , f(δ-int A) $\subseteq$δ-int (f(δ-int (A)) $\subseteq$ δ-int (f(A)).

Now; to prove f is δ-open function.
Let U be δ-open set in X. then δ-int U=U; so f (δ-int U) =f (U);but δ-int (f(U)) $\subseteq$ f(U);therefore f(U) $\subseteq$ f (δ-int U) $\subseteq$ δ-int (f(U)),thus f(U) $\subseteq$ δ-int (f(U)) then f(U) is δ-open set ,so f is δ-open function.

Remark (2.8):

1. If A is open set then A is weakly δ-open set
2. If A is δ-open set then A is weakly δ-open
3. The concept of super continuous and continuous is independent.

Example (1) let X ={a, b, c}, τ={ϕ, X, {b}}; τ'={ϕ, X, {b}, {b, c}} ,Y {1,2,3};σ = {ϕ, Y, {2}}; σ' = {ϕ, Y, {2}, {2,3} }
Let f: $(X,\tau,\tau')\to(Y,\sigma,\sigma')$ such that f(a)=1,f(b)=2,f(c)=3 f is continuous function but it is not super continuous function

Example(2)letX={1,2,3};τ={ϕ,X,{1},{2},{1,2}};τ'={ϕ,X,{1},{2,3}}
Let Y = {a, b, c};σ = {ϕ, Y, {a}, {b}, {a, b}}; σ' = {ϕ, Y, {a}, {b, c}}
Let f: $(X,\tau,\tau')\to(Y,\sigma,\sigma')$ such that :
f(1)=b, f(2)=a ,f(3)=c
It is clear that f is super continuous function but it is continuous function.

4) If f is continuous function then f is w-super continuous function.
5) If f is super continuous then f is w-super continuous function.
6) If f is w- δ-continuous then f is w-super –continuous function

Three converses of (4), (5), and (6) is not true in general. Example (1) and (2) explain that.

7) If f is w- δ-open function then f is almost w-δ-open If f is w- δ-open function

8) If f is always open function then f is almost w-δ-open If f is w- δ-open function

9) If f is always open function then f is open function

10) If f is open function then f is almost W-δ-open function

The converse of (7), (8), (9), (10) is not true in general. The following examples explain that

Example:let$X=\{1,2,3\};\tau=\{\phi,X\};\tau'=\{\phi, X\}$ Let $Y=\{a,b,c\}$, };$\sigma = \{ \phi, Y ,\{a\} ,\{b\},\{a,b\} \}$; $\sigma' = \{ \phi, Y ,\{a\},\{b\},\{a,b\} \}$ Let $f : (X, \tau,\tau') \rightarrow (Y,\sigma,\sigma')$ such that $f(a)=1,f(b)=2,f(c)=3$ f is almost w- δ-open but it is not w- δ-open function and f is not almost open function

In example (2); we note that f is w- δ-open but it is not open function
The following two diagrams are declare the relation among these functions

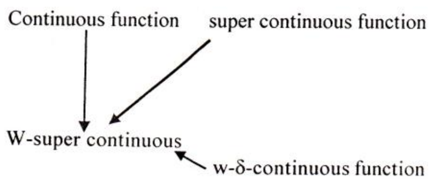

Diagram (1)

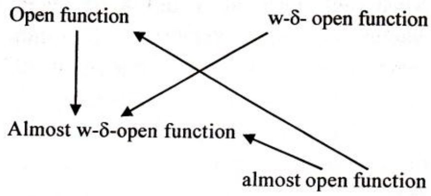

Diagram (2)

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